

# Self-force as probe of internal structure

Soichiro Isoyama<sup>1,2</sup> and Eric Poisson<sup>1</sup>

<sup>1</sup> Department of Physics, University of Guelph, Guelph, Ontario, N1G 2W1, Canada

<sup>2</sup> Yukawa Institute for Theoretical Physics, Kyoto university, Kyoto, 606-8502, Japan

E-mail: isoyama@yukawa.kyoto-u.ac.jp

E-mail: epoisson@uoguelph.ca

**Abstract.** The self-force acting on a (scalar or electric) charge held in place outside a massive body contains information about the body's composition, and can therefore be used as a probe of internal structure. We explore this theme by computing the (scalar or electromagnetic) self-force when the body is a spherical ball of perfect fluid in hydrostatic equilibrium, under the assumption that its rest-mass density and pressure are related by a polytropic equation of state. The body is strongly self-gravitating, and all computations are performed in exact general relativity. The dependence on internal structure is best revealed by expanding the self-force in powers of  $r_0^{-1}$ , with  $r_0$  denoting the radial position of the charge outside the body. To the leading order, the self-force scales as  $r_0^{-3}$  and depends only on the square of the charge and the body's mass; the leading self-force is universal. The dependence on internal structure is seen at the next order,  $r_0^{-5}$ , through a structure factor that depends on the equation of state. We compute this structure factor for relativistic polytropes, and show that for a fixed mass, it increases linearly with the body's radius in the case of the scalar self-force, and quadratically with the body's radius in the case of the electromagnetic self-force. In both cases we find that for a fixed mass and radius, the self-force is smaller if the body is more centrally dense, and larger if the mass density is more uniformly distributed.

PACS numbers: 04.20.-q, 04.40.-b, 04.40.Nr., 41.20.Cv

## 1. Introduction and summary

An electric charge held in place in the curved spacetime of a massive body produces an electric field that responds to the spacetime curvature; a consequence of the interaction is a distortion of the field lines from an otherwise isotropic distribution near the charge, which leads to a net force acting on the particle. This is the physical origin of the electromagnetic self-force [1], a subtle effect that is sensitive to the geometry of spacetime not just in the vicinity of the charge, but everywhere.

Self-force effects in curved spacetime have been vigorously explored; for an extensive review, see Ref. [2]. Most of the recent activity was focused on the gravitational self-force, in an effort to model the inspiral and gravitational-wave emissions of a binary system with a small mass ratio [3, 4, 5]. The prototypical problem, however, goes back to 1980, when Smith and Will [6] calculated the self-force acting on a particle with electric charge  $e$  held in place at a radius  $r_0$  in the

Schwarzschild spacetime of a nonrotating black hole of mass  $M$ . The analogous problem of the self-force acting on a scalar charge  $q$  in the same spacetime was investigated by Wiseman [7]. The results are interesting: The radial component of the electromagnetic self-force (in Schwarzschild coordinates) is equal to  $e^2 M f_0^{1/2} / r_0^3$ , in which  $f_0 := 1 - 2M/r_0$ , while the self-force vanishes in the case of a scalar charge. The electromagnetic self-force is repulsive, meaning that the external force required to keep the particle in place is smaller when the particle is charged, compared to what it would be in the case of a neutral particle. These results are not intuitive: It is not obvious why the electromagnetic self-force should be repulsive, and it is not obvious why the scalar self-force should vanish; for an intriguing explanation based on Newton's third law, refer to Sec. IV of Ref. [8].

The issue that interests us in this paper is the way in which the internal composition of the massive body affects the self-force acting on an electric or scalar charge. Because the self-force is the result of an interaction between the field and the spacetime curvature, and because this interaction will depend on the details of internal structure, it is expected that the self-force will reflect these details, and will therefore reveal some aspects of the body's composition. Our goal is to describe which aspects of the internal structure can be revealed by the self-force.

This line of enquiry was initiated by Burko, Liu, and Soen [8]. Building on earlier work by Unruh [9], they calculated the self-force acting on electric and scalar charges held in place in the exterior spacetime of a massive thin shell. They found that their results agreed with the black-hole results to leading order in an expansion of the self-force in powers of  $r_0^{-1}$ , but that structure-dependent terms appeared at higher-order. The cases of an electric charge outside a body of constant density, and of an electric charge outside a massive conductor, were examined by Shankar and Whiting [10], who confirmed the agreement with the black-hole results for large values of  $r_0$ .

The enquiry was recently pursued by Drivas and Gralla [11], who considered static charges in the exterior spacetime of a massive body of arbitrary composition. They firmly established the universality of the self-force at order  $r_0^{-3}$ , and revealed a dependence on internal structure at order  $r_0^{-5}$  and beyond. More precisely stated, Drivas and Gralla found that when the self-force is expanded in powers of  $r_0^{-1}$ , the leading term that appears at order  $r_0^{-3}$  is independent of the body's internal structure. In the case of an electric charge, it is always equal to the Smith-Will result  $e^2 M / r_0^3$ , and in the case of a scalar charge, it is always equal to the vanishing Wiseman result. (This last statement holds for a scalar field that is minimally coupled to the spacetime curvature. For nonminimal coupling, the self-force continues to vanish when the charge is held outside a black hole, but it is equal to  $2\xi q^2 M / r_0^3$  when there is a material body, where  $\xi$  is a dimensionless coupling constant. In this case the universality of the scalar self-force at order  $r_0^{-3}$  partially breaks down: the self-force is only insensitive to the details of internal structure when the body is not a black hole.)

The main concern of Drivas and Gralla was to display the universality of the self-force at order  $r_0^{-3}$ , and their other discovery, that there is a dependence on internal structure at order  $r_0^{-5}$  and beyond, was left largely unexplored. This is the theme that we intend to pursue here: How does the self-force depend on the body's internal structure, and which aspects of the internal composition can be inferred from a close examination of the self-force? These issues are interesting, because it is very difficult, in general relativity, to collect any information about a body's internal structure when the body is spherical, and when one is limited to external measurements. It is impossible, for example, to reveal any aspect of the internal structure by measuring

the motion of a test body. Going beyond test bodies, for example by measuring the orbital motion of a two-body system of comparable masses, reveals very little: the strong formulation of the principle of equivalence implies that the orbital motion is necessarily insensitive to the details of internal structure, until the bodies are so close together that significant tidal deformations are generated [12, 13, 14, 15, 16, 17].

The self-force, on the other hand, can be used as a probe of internal structure. Indeed, one can easily imagine a thought experiment designed to exploit the self-force in this manner. Suppose that one is given a neutral particle, a second particle with an electric charge  $e$  (or a scalar charge  $q$ ), and a massless rope to keep each particle in place in the body's gravitational field; the other end of the rope is held at a safe distance by an external agent. Keeping the neutral particle at a position  $r_0$  requires the external agent to hold the rope firmly, and the tension in the rope reveals the body's gravitational force on the particle. Replacing the neutral particle by the charge produces a slight deficit in the tension, which corresponds to the repulsive action of the self-force; the self-force at position  $r_0$  can thus be measured. Repeating the measurement for many positions allows one to infer the self-force as a function of  $r_0$ , together with its expansion in powers of  $r_0^{-1}$ ; the terms of order  $r_0^{-5}$  and beyond are those that reveal aspects of the body's internal structure. There is, of course, nothing practical about this. But it is of interest to observe that as a matter of principle, one can infer elusive details of internal structure by careful measurements of a self-force acting on a static charge.

To investigate the dependence of the self-force on the internal structure of a massive body, we consider a spherical star that consists of a perfect fluid in hydrostatic equilibrium. The fluid is characterized by a (baryonic) rest-mass density  $\rho$  and a pressure  $p$ , and these variables are taken to be related by the polytropic equation of state  $p = K\rho^{1+1/n}$ , in which  $K$  is a scaling constant, and  $n$  is the polytropic index (another constant). The star possesses a mass  $M$  and a radius  $R$ , and a (scalar or electric) charge is placed at a radius  $r_0 > R$  in the vacuum region exterior to the star.

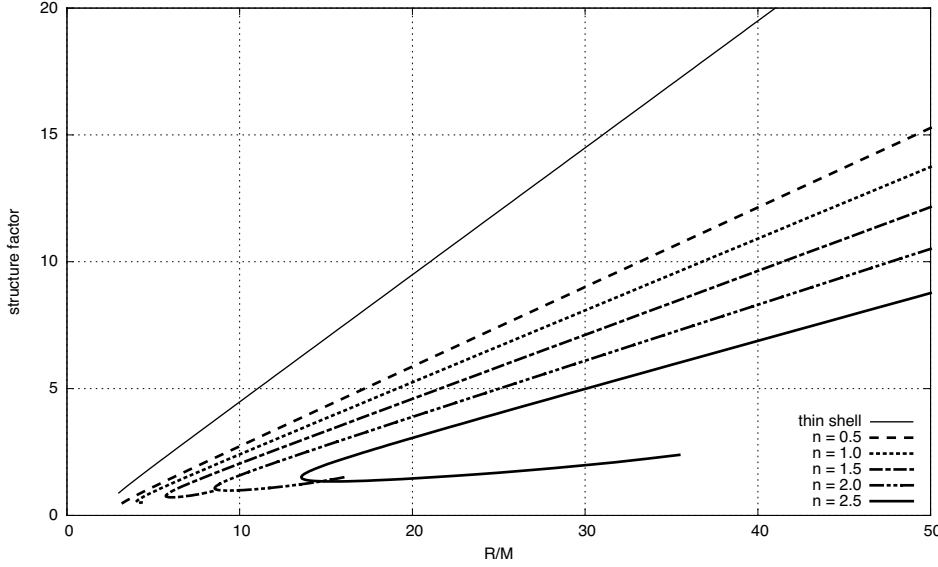
Following Drivas and Gralla [11], we compute the difference between two self-forces. The first is the actual self-force acting on the charge held in place outside the polytropic star, and the second is the self-force that would be acting if the charge were instead situated outside a Schwarzschild black hole; the bodies have the same mass, and the charge is placed at the same position outside each body. It is this difference that contains the dependence of the self-force on the body's internal structure. We show below that in the case of a scalar charge, the difference can be expressed as

$$\Delta F_{\text{scalar}}^r = -\left(\frac{q}{M}\right)^2 \left(\frac{z_0 - 1}{z_0 + 1}\right)^{3/2} \sum_{\ell=1}^{\infty} (2\ell + 1) S_{\ell}^{\text{scalar}} Q_{\ell}(z_0) Q'_{\ell}(z_0), \quad (1.1)$$

in which  $z_0 := r_0/M - 1$ ,  $Q_{\ell}(z_0)$  is a Legendre function of the second kind, a prime indicates differentiation with respect to  $z_0$ , and  $S_{\ell}^{\text{scalar}}$  is a structure factor that depends on the stellar interior; an expression is provided below in Eq. (3.24). In the case of an electric charge we have instead

$$\Delta F_{\text{em}}^r = -\left(\frac{e}{M}\right)^2 \left(\frac{z_0 - 1}{z_0 + 1}\right)^{3/2} \sum_{\ell=1}^{\infty} (2\ell + 1) S_{\ell}^{\text{em}} \left[ Q_{\ell}(z_0) - \frac{(z_0 - 1)Q'_{\ell}(z_0)}{\ell(\ell + 1)} \right] Q'_{\ell}(z_0), \quad (1.2)$$

in which  $S_{\ell}^{\text{em}}$  is the corresponding structure factor for the electromagnetic self-force; see Eq. (4.21) below.



**Figure 1.** Structure factor  $S_1^{\text{scalar}}$  for the scalar self-force, plotted as a function of  $R/M$  for selected polytropic equations of state labelled by the polytropic index  $n$ . The curve for a massive thin shell is also displayed for comparison. At large  $R/M$ , the structure factor increases linearly with  $R/M$ , with a slope that depends on the polytropic index. For some equations of state the structure factor is multi-valued at small  $R/M$ ; the low-lying branch, however, corresponds to stellar configurations beyond the maximum mass, which are dynamically unstable.

When  $r_0 \gg M$  the self-force differences are dominated by the leading term  $\ell = 1$  in the sums, and the expressions reduce to

$$\Delta F_{\text{scalar}}^r \sim \frac{2}{3} S_1^{\text{scalar}} \frac{q^2 M^3}{r_0^5} \quad (1.3)$$

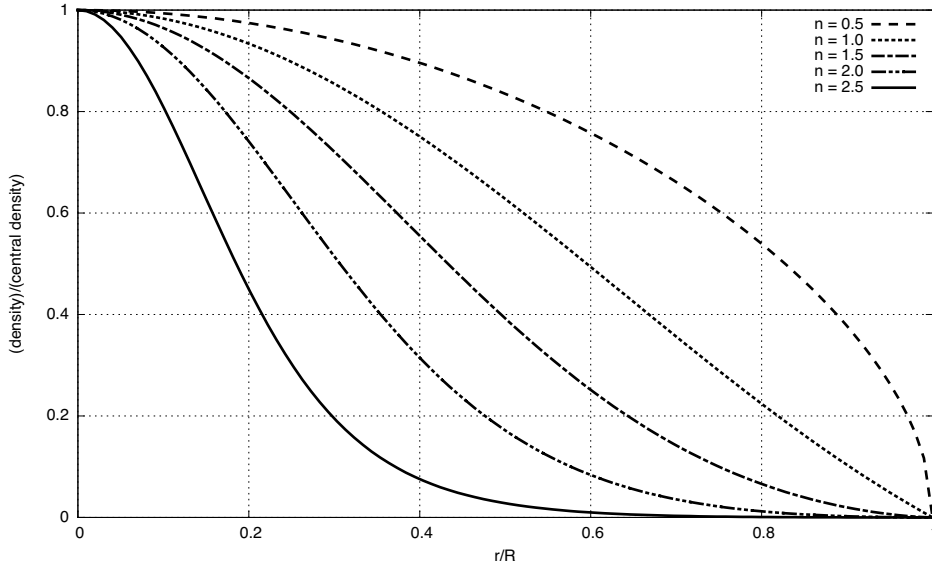
and

$$\Delta F_{\text{em}}^r \sim \frac{4}{3} S_1^{\text{em}} \frac{e^2 M^3}{r_0^5}. \quad (1.4)$$

This is the statement that to leading order in an expansion of the self-force in powers of  $r_0^{-1}$ , the details of internal structure are revealed at order  $r_0^{-5}$ ; and these are contained in the structure factors  $S_1^{\text{scalar}}$  and  $S_1^{\text{em}}$ .

In Fig. 1 we present plots of  $S_1^{\text{scalar}}$  as a function of  $R/M$  for selected values of the polytropic index  $n$ . We find that for large values of  $R/M$ , the structure factor increases linearly with  $R/M$ , with a slope that depends on the polytropic index; these results imply that for fixed  $r_0$  and  $M$ , the self-force increases linearly with the stellar radius. This is the same scaling that was found by Burko, Liu, and Soen [8] in the case of a massive thin shell, for which we also plot the structure factor.

An increase of the self-force with the stellar radius is to be expected. The self-force results from an interaction between the field produced by the scalar charge and the curvature of spacetime. Since the self-force would vanish in the pure Schwarzschild spacetime of a black hole, the interaction is limited to the region of spacetime occupied by the matter. For a fixed  $M$ , the region of interaction increases with the stellar radius



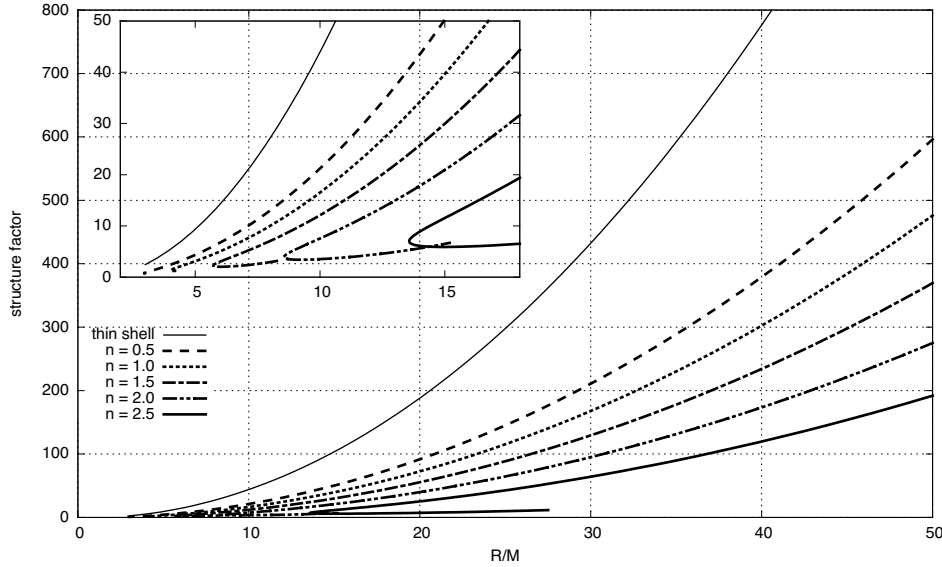
**Figure 2.** Density profiles for selected polytropic models. The mass density  $\rho$ , normalized by the central density  $\rho_c$ , is plotted as a function of  $r/R$ . Each model corresponds to a body of radius-to-mass ratio  $R/M \simeq 15$ , and each curve is labelled by the polytropic index  $n$ . The corresponding values of the relativistic parameter  $b$  (defined in Sec. 2) are  $3.830543764152575e-2$  for  $n = 0.5$ ,  $4.116715459636999e-2$  for  $n = 1.0$ ,  $4.733363787126200e-2$  for  $n = 1.5$ ,  $5.977719194440032e-2$  for  $n = 2.0$ , and  $9.929978563520857e-2$  for  $n = 2.5$ . The figure reveals that as  $n$  increases, the body becomes increasingly centrally dense.

$R$ , and it follows that the self-force should increase with  $R$ . This argument, however, does not explain why the enhancement with size is linear in  $R$  instead of some other relationship. The argument continues to apply to the case of a massive thin shell, for which the curvature is entirely concentrated within the shell; here also the size of the interaction region increases with the shell's radius, leading again to an increase of the self-force.

In addition to the linear scaling of the self-force with  $R/M$ , the curves displayed in Fig. 1 reveal that for a fixed  $R/M$ , the self-force is a decreasing function of the polytropic index. An explanation for this effect can be deduced by extending the preceding argument to take into account the distribution of mass within the star. In Fig. 2 we show density profiles for polytropic models that share the same ratio  $R/M$ ; one sees that models with larger values of  $n$  are more centrally dense than models with smaller values. A centrally-dense star will possess a smaller region of effective interaction with the scalar field than a more uniform star, and this will produce a smaller self-force.

In Fig. 3 we present plots of  $S_1^{\text{em}}$  as a function of  $R/M$  for selected equations of state. We find that for large values of  $R/M$ , the structure factor increases quadratically with  $R/M$ , with a curvature that depends on the polytropic index; these results imply that for fixed  $r_0$  and  $M$ , the self-force increases quadratically with the stellar radius. This is the same scaling that was found by Burko, Liu, and Soen [8] in the case of a massive thin shell.

The increase of the self-force difference with the stellar radius can be explained



**Figure 3.** Structure factor  $S_1^{\text{em}}$  for the electromagnetic self-force, plotted as a function of  $R/M$  for selected polytropic equations of state labelled by the polytropic index  $n$ . The curve for a massive thin shell is also displayed for comparison. At large  $R/M$ , the structure factor increases quadratically with  $R/M$ , with a curvature that depends on the equations of state.

in the same way as for the scalar case. Here the self-force difference results from an interaction between the electromagnetic field and the matter distribution, and a larger star gives rise to a larger region of interaction, and therefore a larger self-force difference; the argument does not explain why the enhancement with size is quadratic in  $R$  in the electromagnetic case. The property that the self-force difference, for a fixed  $R/M$ , is a decreasing function of the polytropic index, is also explained in the same way, with the help of Fig. 2. Here also the property is a consequence of the fact that stellar models with larger values of  $n$  are more centrally dense than models with smaller values.

The main conclusion of this work is that aspects of the internal structure of a spherical star can be revealed in a close examination of the self-force at order  $r_0^{-5}$ . Our computations show that the self-force increases with  $R/M$ , in a linear manner for the scalar case, and in a quadratic manner for the electromagnetic case. In addition, we find that the rate of increase depends on the equation of state, a centrally dense body producing a smaller rate of increase than a more uniform body.

In the remainder of the paper we establish the results presented previously. We begin in Sec. 2 with a description of relativistic polytropes. In Sec. 3 we describe the integration of the scalar-field equation in the spacetime of a massive body, and the computation of the self-force. In Sec. 4 we describe the calculations for the case of an electric charge.

## 2. Polytropic stellar models

In this section we describe the polytropic stellar models that are involved in our examination of the influence of internal structure on the scalar and electromagnetic self-force. Our models are static and spherically symmetric, and we write the spacetime metric as

$$ds^2 = -e^{2\psi} dt^2 + f^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.1)$$

with  $\psi$  a function of  $r$  and  $f := 1 - 2m(r)/r$ . For a perfect fluid at rest the field equations are

$$\frac{dm}{dr} = 4\pi r^2 \sigma, \quad (2.2)$$

$$\frac{d\psi}{dr} = \frac{1}{r^2 f} (m + 4\pi r^3 p), \quad (2.3)$$

$$\frac{dp}{dr} = -(\sigma + p) \frac{d\psi}{dr} = -\frac{\sigma + p}{r^2 f} (m + 4\pi r^3 p), \quad (2.4)$$

in which  $\sigma$  is the energy density and  $p$  the pressure; the last equation is the condition of hydrostatic equilibrium.

The polytropic equations of state are

$$p = K \rho^{1+1/n}, \quad \epsilon = np, \quad (2.5)$$

in which  $K$  and  $n$  are constants,  $\rho$  is the rest-mass density, and  $\epsilon$  the internal thermodynamic energy; the total energy density is then  $\sigma = \rho + \epsilon$ .

To integrate the field equations we introduce the scaling quantities  $\rho_c$  (the central mass density),  $p_c := K \rho_c^{1+1/n}$  (the central pressure),  $m_0 := 4\pi \rho_c r_0^3$  (a mass scale), and the squared length scale  $r_0^2 := (n+1)p_c/(4\pi \rho_c^2)$ . We introduce also the dimensionless relativistic parameter

$$b := \frac{p_c}{\rho_c} = K \rho_c^{1/n}, \quad (2.6)$$

in terms of which  $m_0/r_0 = (n+1)p_c/\rho_c = (n+1)b$ . We make use of the dimensionless variables  $\xi$ ,  $\theta$ , and  $\mu$  such that  $r = r_0 \xi$ ,  $\rho = \rho_c \theta^n$ , and  $m = m_0 \mu$ . In terms of the Lane-Emden variable  $\theta$  we also have  $p = p_c \theta^{n+1}$ ,  $\epsilon = np_c \theta^{n+1}$ , and  $\sigma = \rho_c (1 + nb\theta) \theta^n$ .

The field equations become

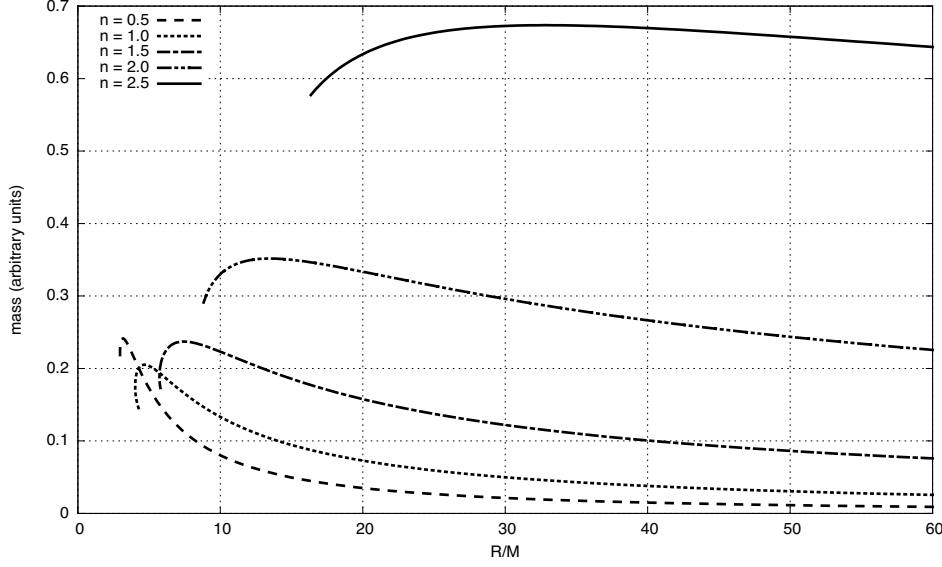
$$\frac{d\mu}{d\xi} = \xi^2 (1 + nb\theta) \theta^n, \quad (2.7)$$

$$\frac{d\theta}{d\xi} = -\frac{1}{\xi^2 f} [1 + (n+1)b\theta] (\mu + b\xi^3 \theta^{n+1}), \quad (2.8)$$

$$\frac{d\psi}{d\xi} = \frac{(n+1)b}{\xi^2 f} (\mu + b\xi^3 \theta^{n+1}), \quad (2.9)$$

with  $f = 1 - 2(n+1)b\mu/\xi$ . The equations are integrated outward from  $\xi = 0$ , at which we impose the boundary conditions  $\mu = 0$ ,  $\theta = 1$ , and  $\psi = \psi_c$ . Integration stops at  $\xi = \xi_1$ , at which  $\mu = \mu_1$ ,  $\theta = 0$ , and  $\psi = \psi_1$ . The star's radius is then  $R = r_0 \xi_1$ , its mass is  $M = m_0 \mu_1$ , and the value of  $\psi_c$  is chosen so that  $e^{2\psi_1} = 1 - 2M/R$ , to ensure that the solution joins smoothly to the Schwarzschild metric when  $r > R$ .

The numerical integration of the equations is facilitated by using  $\nu := \mu/\xi^3$  as a substitute for the mass function, and  $x := \ln \xi$  as a substitute for the radial variable.



**Figure 4.** Mass of a relativistic polytrope, in units of  $\bar{m}_0$ , as a function of  $R/M$ , for selected values of the polytropic index  $n$ . The lowest curve corresponds to  $n = 0.5$ , higher curves correspond to increasing values of  $n$ , and the highest curve corresponds to  $n = 2.5$ . In each case the mass reaches a maximum value at a minimum value of  $R/M$ . Configurations with smaller  $R/M$  are dynamically unstable and therefore unphysical.

In terms of these we have

$$\frac{d\nu}{dx} = (1 + nb\theta)\theta^n - 3\nu, \quad (2.10)$$

$$\frac{d\theta}{dx} = -\frac{\xi^2}{f} [1 + (n+1)b\theta] (\nu + b\theta^{n+1}), \quad (2.11)$$

$$\frac{d\psi}{dx} = (n+1)b\frac{\xi^2}{f} (\nu + b\theta^{n+1}), \quad (2.12)$$

with  $f = 1 - 2(n+1)b\xi^2\nu$  and  $\xi = e^x$ . The boundary conditions are now placed at  $x = -\infty$ , at which  $\nu = \frac{1}{3}(1 + nb)$ ,  $\theta = 1$ , and  $\psi = \psi_c$ . In practice the integrations are started at  $x = x_0 < 0$  such that  $\xi_0 = e^{x_0}$  is very small, and starting values for  $\nu$ ,  $\theta$ , and  $\psi$  are obtained by expanding each quantity in powers of  $\xi^2$  and determining the coefficients with the help of the differential equations.

Integration of the field equations for a selected value of the polytropic index  $n$  gives rise to a continuous sequence of stellar models parameterized by  $b$ , which acts as a substitute for the central density  $\rho_c$ . Because  $m_0$  and  $r_0$  depend on the central density (and therefore on  $b$ ), it is necessary to rescale the mass and length units so as to eliminate this dependence before producing plots of the mass  $M$  and radius  $R$  as functions of  $b$  on the sequence. We therefore set

$$M = \bar{m}_0 b^{(3-n)/2} \mu_1, \quad R = \bar{r}_0 b^{(1-n)/2} \xi_1, \quad (2.13)$$

in which

$$\bar{m}_0 := m_0 b^{-(3-n)/2} = \frac{(n+1)^{3/2} K^{n/2}}{(4\pi)^{1/2}}, \quad (2.14)$$



$$\bar{r}_0 := r_0 b^{-(1-n)/2} = \frac{(n+1)^{1/2} K^{n/2}}{(4\pi)^{1/2}} \quad (2.15)$$

are the rescaled mass and length units, respectively, which are independent of  $b$  and therefore constant on the sequence. Plots of  $M/\bar{m}_0$  as a function of  $R/M$  for selected values of the polytropic index  $n$  are presented in Fig. 4.

### 3. Scalar self-force

#### 3.1. Scalar field and self-force in spherical spacetimes

We consider the self-force acting on a static scalar charge  $q$  moving on a world line  $c$  in a curved spacetime; the world line is described by the parametric equations  $x^\alpha = z^\alpha(\tau)$ , in which  $\tau$  is proper time. The charge produces a scalar potential  $\Phi$  that satisfies the wave equation

$$\square \Phi = -4\pi q \int_c \delta_4(x, z(\tau)) d\tau, \quad (3.1)$$

in which  $\square := g^{\alpha\beta} \nabla_\alpha \nabla_\beta$  is the covariant wave operator, and  $\delta_4(x, z)$  is a scalarized Dirac distribution. The self-force acting on the scalar charge is given by

$$F^\alpha = q(g^{\alpha\beta} + u^\alpha u^\beta) \nabla_\beta \Phi^R, \quad (3.2)$$

in which  $u^\alpha := dz^\alpha/d\tau$  is the charge's velocity vector, and  $\Phi^R := \Phi - \Phi^S$  is the difference between  $\Phi$  and the Detweiler-Whiting singular potential  $\Phi^S$ [18]; the regular potential  $\Phi^R$  is known to be smooth on  $c$ , and to be solely responsible for the self-force.

For our particular application we take the spacetime to be static and spherically symmetric, we write the metric as in Eq. (2.1), and we choose  $c$  to be a static world line at  $r = r_0 > R$ . We decompose the potential in spherical harmonics,

$$\Phi(r, \theta, \phi) = \sum_{\ell m} \Phi_{\ell m}(r) Y_{\ell m}(\theta, \phi), \quad (3.3)$$

and substitution within the wave equation produces

$$r^2 \frac{d^2 \Phi_{\ell 0}}{dr^2} + \left( 2 + \frac{r}{2f} \frac{df}{dr} + r \frac{d\psi}{dr} \right) r \frac{d\Phi_{\ell 0}}{dr} - \frac{\ell(\ell+1)}{f} \Phi_{\ell 0} = -4\pi q \sqrt{\frac{2\ell+1}{4\pi}} f_0^{-1/2} \delta(r-r_0), \quad (3.4)$$

in which  $f_0 := f(r_0)$ . Without loss of generality we placed the charge on the axis  $\theta = 0$ , and exploited the property  $Y_{\ell m}(0, \phi) = \sqrt{(2\ell+1)/(4\pi)} \delta_{m,0}$  of spherical-harmonic functions. The modes  $m \neq 0$  of the scalar potential vanish.

It is easy to show that the time and angular components of the self-force vanish in a static and spherically-symmetric situation, and that the radial component is given by

$$F^r = q f_0 \lim_{x \rightarrow z} \sum_\ell \left[ (\partial_r \Phi)_\ell - (\partial_r \Phi^S)_\ell \right], \quad (3.5)$$

in which

$$(\partial_r \Phi)_\ell := \sum_{m=-\ell}^{\ell} \frac{d\Phi_{\ell m}}{dr}(r) Y_{\ell m}(\theta, \phi) \quad (3.6)$$

are the *multipole coefficients* of  $\partial_r \Phi$ , while  $(\partial_r \Phi^S)_\ell$  are those of the singular potential  $\Phi^S$ . The limit in Eq. (3.5) can be taken by setting  $r = r_0 + \Delta$ ,  $\theta = 0$ , and letting  $\Delta \rightarrow 0$  (from either direction). With this choice, we find that

$$(\partial_r \Phi)_\ell = \sqrt{\frac{2\ell+1}{4\pi}} \frac{d\Phi_{\ell 0}}{dr}(r_0 + \Delta). \quad (3.7)$$

The multipole coefficients can therefore be obtained from the solution to Eq. (3.4). This equation must be integrated in the stellar interior of the polytropic model, and in the body's Schwarzschild exterior.

### 3.2. Interior

The equation satisfied by the  $\Phi_{\ell 0}$  mode of the scalar potential was given above in Eq. (3.4), and since the charge is situated in the exterior portion of the spacetime ( $r_0 > R$ ), the right-hand side of the equation is zero everywhere within the interior. To integrate the equation we implement a change of variables from  $\Phi_{\ell 0}$  to  $\eta_\ell$  defined by

$$\eta_\ell := \frac{d \ln \Phi_{\ell 0}}{d \ln r}. \quad (3.8)$$

The equation for  $\Phi_{\ell 0}$  becomes

$$\xi \frac{d\eta_\ell}{d\xi} + \eta_\ell(\eta_\ell - 1) + \mathcal{D}\eta_\ell - \frac{\ell(\ell+1)}{f} = 0 \quad (3.9)$$

when expressed in terms of  $\eta_\ell$  and  $\xi := r/r_0$ , with  $\mathcal{D} := 2 + \frac{1}{2}f^{-1}r(df/dr) + r(d\psi/dr)$  given explicitly by

$$\mathcal{D} = \frac{1}{f} \left[ 2 - (n+1)b\xi^2 \left\{ 2\nu + [1 + (n-1)b\theta]\theta^n \right\} \right]. \quad (3.10)$$

The equation is integrated outward from  $\xi = 0$ , at which  $\eta_\ell = \ell$ . Here also the numerical treatment can be improved by adopting  $x = \ln \xi$  as a new independent variable, and adding a few more terms to an expansion of  $\eta_\ell$  in powers of  $\xi^2$ . Integration proceeds until  $\xi = \xi_1$ , at which  $\eta_\ell = \eta_\ell^1$ . As we shall see, the number  $\eta_\ell^1$  is the entirety of the information required about the internal solution.

### 3.3. Exterior

Outside the matter the metric is described by the Schwarzschild solution, so that  $e^{2\psi} = f = 1 - 2M/r$ . Equation (3.4) becomes

$$r^2 f \frac{d^2 \Phi}{dr^2} + 2 \left( 1 - \frac{M}{r} \right) r \frac{d\Phi}{dr} - \ell(\ell+1)\Phi = -4\pi q \sqrt{\frac{2\ell+1}{4\pi}} f_0^{1/2} \delta(r - r_0), \quad (3.11)$$

where we suppress the label “ $\ell 0$ ” on  $\Phi$ . It is helpful to use  $z := r/M - 1$  instead of  $r$  as a radial variable, and to re-express the differential equation as

$$(z^2 - 1) \frac{d^2 \Phi}{dz^2} + 2z \frac{d\Phi}{dz} - \ell(\ell+1)\Phi = -4\pi \frac{q}{M} \sqrt{\frac{2\ell+1}{4\pi}} \left( \frac{z_0 - 1}{z_0 + 1} \right)^{1/2} \delta(z - z_0), \quad (3.12)$$

where  $z_0 := r_0/M - 1$ . Away from  $z_0$  the right-hand side of the equation vanishes, and the solution is a linear superposition of Legendre functions  $P_\ell(z)$  and  $Q_\ell(z)$ . The

presence of the delta function implies that the solution must respect the junction conditions

$$[\Phi] = 0, \quad \left[ \frac{d\Phi}{dz} \right] = -4\pi \frac{q}{M} \sqrt{\frac{2\ell+1}{4\pi}} \frac{1}{(z_0-1)^{1/2}(z_0+1)^{3/2}} \quad (3.13)$$

at  $z = z_0$ ; here  $[f] := f(z = z_0 + \epsilon) - f(z = z_0 - \epsilon)$ , with  $\epsilon$  approaching zero from above, is the jump of the function  $f$  across  $z = z_0$ .

### 3.4. Matching

The interior and exterior solutions must match smoothly at  $r = R$ , or  $z = Z := R/M - 1$ , and the two exterior solutions (for  $z < z_0$  and  $z > z_0$ ) must satisfy the junction conditions at  $z = z_0$ . For  $z < Z$  we have that  $\Phi = \Phi_{\text{in}}$ , for  $Z < z < z_0$  we have that  $\Phi = AP_\ell(z) + BQ_\ell(z)$ , where  $A$  and  $B$  are coefficients to be determined, and for  $z > z_0$  we have that  $\Phi = CQ_\ell(z)$ , where  $C$  is a third unknown coefficient; we exclude a term in  $P_\ell(z)$  when  $z > z_0$  because it would produce a diverging field at  $z = \infty$ . Letting  $\alpha := AP_\ell(z_0)$ ,  $\beta := BQ_\ell(z_0)$ ,  $\gamma := CQ_\ell(z_0)$ , and  $\delta := \Phi_{\text{in}}(Z)$ , we find that the matching conditions give rise to the relations

$$\alpha = (z_0 - 1)(z_0 + 1)P_\ell(z_0)Q_\ell(z_0)J, \quad (3.14)$$

$$\beta = -(z_0 - 1)(z_0 + 1) \frac{(Z + 1)P'_\ell(Z) - \eta_\ell^1 P_\ell(Z)}{(Z + 1)Q'_\ell(Z) - \eta_\ell^1 Q_\ell(Z)} [Q_\ell(z_0)]^2 J, \quad (3.15)$$

in which

$$J := 4\pi \frac{q}{M} \sqrt{\frac{2\ell+1}{4\pi}} \frac{1}{(z_0-1)^{1/2}(z_0+1)^{3/2}}, \quad (3.16)$$

along with  $\gamma = \alpha + \beta$  and  $\delta = [P_\ell(Z)/P_\ell(z_0)]\alpha + [Q_\ell(Z)/Q_\ell(z_0)]\beta$ . To arrive at these results we used the Wronskian property of the Legendre functions,  $P_\ell Q'_\ell - Q_\ell P'_\ell = -(z^2 - 1)^{-1}$ , with a prime indicating differentiation with respect to  $z$ . As we shall see, the quantity of prime interest for a self-force computation is  $\gamma$ .

### 3.5. Black-hole problem

If the scalar charge were placed outside a black hole instead of in the exterior region of a spherical star, the Schwarzschild metric would apply everywhere instead of being restricted to  $r > R$  or  $z > Z$ . In such a case there would be no interior solution to the scalar-field equation, and the two external solutions would be given by  $\Phi = \bar{A}P_\ell(z)$  for  $z < z_0$ , and  $\Phi = \bar{C}Q_\ell(z)$  for  $z > z_0$ ; a term in  $Q_\ell(z)$  is excluded when  $z < z_0$  because it would diverge on the event horizon. Letting  $\bar{\alpha} := \bar{A}P_\ell(z_0)$  and  $\bar{\gamma} := \bar{C}Q_\ell(z_0)$ , we find that the junction conditions at  $z = z_0$  produce  $\bar{\alpha} = \bar{\gamma} = \alpha$ .

### 3.6. Self-force difference

The radial component of the self-force acting on a scalar charge  $q$  at  $r = r_0$  is given by Eq. (3.5), which we express as

$$F_{\text{star}}^r = qf_0 \sum_{\ell} \left[ (\partial_r \Phi_{\text{star}})_\ell - (\partial_r \Phi^S)_\ell \right], \quad (3.17)$$

with the understanding that the right-hand side is evaluated in the limit  $\Delta \rightarrow 0$ .

In addition to this self-force, we may also consider the self-force acting on another scalar charge  $q$  placed at  $r = r_0$  in the pure Schwarzschild spacetime of a black hole. For this situation we would have instead

$$F_{\text{hole}}^r = qf_0 \sum_{\ell} \left[ (\partial_r \Phi_{\text{hole}})_{\ell} - (\partial_r \Phi^S)_{\ell} \right], \quad (3.18)$$

which refers to  $\Phi_{\text{hole}}$ , the scalar potential as computed in the pure Schwarzschild spacetime, and *to the same singular potential* as in Eq. (3.17). The singular potential is the same in each case, because it depends on the structure of spacetime only in the immediate vicinity of the charge, where it is described by the Schwarzschild metric.

Taking the difference between Eqs. (3.17) and (3.18), we have that

$$\Delta F^r := F_{\text{star}}^r - F_{\text{hole}}^r = qf_0 \sum_{\ell} \left[ (\partial_r \Phi_{\text{star}})_{\ell} - (\partial_r \Phi_{\text{hole}})_{\ell} \right], \quad (3.19)$$

and we see that the *difference* between the two self-forces can be computed without the involvement of the singular field; this method of regularization was originally devised by Drivas and Gralla [11]. Because the black-hole force actually vanishes [7], we have that

$$F_{\text{star}}^r = \Delta F^r, \quad (3.20)$$

and  $\Phi_{\text{hole}}$  can in fact be identified with the Detweiler-Whiting singular potential.

The force difference can now be expressed as

$$\Delta F^r = qf_0 \sum_{\ell=0}^{\infty} \sqrt{\frac{2\ell+1}{4\pi}} \left[ \frac{d\Phi_{\text{star}}}{dr}(r_0 + \Delta) - \frac{d\Phi_{\text{hole}}}{dr}(r_0 + \Delta) \right], \quad (3.21)$$

in which each  $\Phi$  stands for the  $\ell 0$  mode of the spherical-harmonic expansion. While each individual mode-sum for  $F_{\text{star}}^r$  and  $F_{\text{hole}}^r$  would fail to converge, the mode-sum for  $\Delta F^r$  can be shown to converge exponentially.

The mode-sum requires the evaluation of  $d\Phi_{\text{star}}/dr$  and  $d\Phi_{\text{hole}}/dr$  at  $r = r_0 + \Delta$ . It is convenient to take  $\Delta = 0^+$ , so that  $r = r_0$  is approached from above. We therefore insert  $Md\Phi_{\text{star}}/dr = \gamma Q'_{\ell}(z_0)/Q_{\ell}(z_0)$  and  $Md\Phi_{\text{hole}}/dr = \bar{\gamma} Q'_{\ell}(z_0)/Q_{\ell}(z_0)$ , and obtain

$$\Delta F^r = \frac{q}{M} f_0 \sum_{\ell=1}^{\infty} \sqrt{\frac{2\ell+1}{4\pi}} (\gamma - \bar{\gamma}) \frac{Q'_{\ell}(z_0)}{Q_{\ell}(z_0)}; \quad (3.22)$$

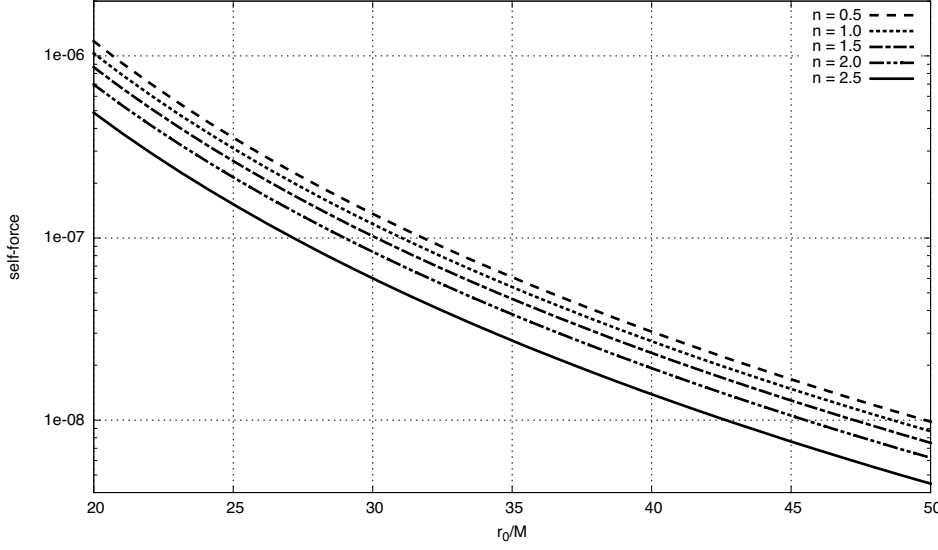
the sum now excludes  $\ell = 0$ , because the monopole solutions are the same in the two spacetimes and therefore do not contribute to the difference. (The monopole solution describes the field of a spherical shell of charge  $q$  at  $r = r_0$ ; in both spacetimes the field vanishes inside the shell, and goes as  $q/r^2$  outside the shell.) Inserting the result obtained previously for  $\gamma - \bar{\gamma} = \beta$ , we arrive at the expression of Eq. (1.1),

$$\Delta F_{\text{star}} = - \left( \frac{q}{M} \right)^2 \left( \frac{z_0 - 1}{z_0 + 1} \right)^{3/2} \sum_{\ell=1}^{\infty} (2\ell + 1) S_{\ell} Q_{\ell}(z_0) Q'_{\ell}(z_0), \quad (3.23)$$

in which

$$S_{\ell} := \frac{(Z + 1)P'_{\ell}(Z) - \eta_{\ell}^1 P_{\ell}(Z)}{(Z + 1)Q'_{\ell}(Z) - \eta_{\ell}^1 Q_{\ell}(Z)} \quad (3.24)$$

is a structure factor that depends on the stellar model (through the stellar radius  $R$  and the interior constant  $\eta_{\ell}^1$ ) but is independent of  $r_0$ . We recall that  $z := r/M - 1$ ,  $Z := R/M - 1$ ,  $z_0 := r_0/M - 1$ , and that a prime indicates differentiation with respect



**Figure 5.** Scalar self-force difference  $\Delta F^r$ , in units of  $(q/M)^2$ , plotted as a function of  $r_0/M$  for selected polytropic equations of state labelled by the polytropic index  $n$ . Each stellar model corresponds to a body of radius-to-mass ratio  $R/M \simeq 15$  (see Fig 2). The falloff with  $r_0$  is consistent with the asymptotic behaviour  $\sim r_0^{-5}$  predicted by Eq. (1.3), and for fixed  $r_0/M$  the self-force difference is seen to decrease with increasing  $n$ ; this behaviour is discussed and explained in Sec. 1.

to  $z$ . The mode-sum can be evaluated straightforwardly once the numerical results for  $\eta_\ell^1$  and  $R/M$  are available.

The self-force of Eq. (3.23) can be computed as a function of  $r_0$  for selected stellar models characterized by a polytropic index  $n$  and a relativistic parameter  $b$ . Sample results are displayed in Fig. 5, and our findings were presented more fully in Sec. 1. The self-force for a massive thin shell can be computed by integrating the scalar-field equation in the flat interior and matching the solution to the Schwarzschild exterior; this yields  $\eta_\ell^1 = \ell/\sqrt{1-2M/R}$ , which can be inserted within Eq. (3.24) and shown to reproduce the results of Burko, Liu, and Soen [8].

## 4. Electromagnetic self-force

### 4.1. Electromagnetic field and self-force in spherical spacetimes

We next consider the self-force acting on a static electric charge  $e$  moving on a world line  $c$  in a curved spacetime. The charge produces a vector potential  $\Phi_\alpha$  that satisfies the wave equation

$$\square \Phi_\alpha - R_\alpha^\beta \Phi_\beta = -4\pi e \int_c u^\alpha \delta_4(x, z) d\tau \quad (4.1)$$

in the Lorenz gauge  $\nabla^\alpha \Phi_\alpha = 0$ ; here  $R_{\alpha\beta}$  is the spacetime's Ricci tensor. The potential gives rise to the electromagnetic field  $F_{\alpha\beta} = \nabla_\alpha \Phi_\beta - \nabla_\beta \Phi_\alpha$ , and the self-force acting on the electric charge is given by

$$F^\alpha = e F_{\alpha\beta} u^\beta, \quad (4.2)$$

in which  $F_{\mathbf{R}\beta}^\alpha := F_\beta^\alpha - F_{\mathbf{S}\beta}^\alpha$  is the difference between the actual electromagnetic field and the Detweiler-Whiting singular field [18]; the regular field  $F_{\mathbf{R}\beta}^\alpha$  is known to be smooth on  $c$ , and to be solely responsible for the self-force.

For our particular application we take the spacetime to be static and spherically symmetric, we write the metric as in Eq. (2.1), and we choose  $c$  to be a static world line at  $r = r_0 > R$ . The only relevant component of the vector potential is  $\Phi_t$ , and we decompose it as

$$\Phi_t(r, \theta, \phi) = \sum_{\ell m} \Phi_{t\ell m}(r) Y_{\ell m}(\theta, \phi). \quad (4.3)$$

Substitution within the wave equation produces

$$\begin{aligned} r^2 \frac{d^2 \Phi_{t\ell 0}}{dr^2} + \left( 2 + \frac{r}{2f} \frac{df}{dr} - r \frac{d\psi}{dr} \right) r \frac{d\Phi_{t\ell 0}}{dr} - \frac{\ell(\ell+1)}{f} \Phi_{t\ell 0} \\ = 4\pi e \sqrt{\frac{2\ell+1}{4\pi}} e^{\psi_0} f_0^{-1/2} \delta(r - r_0), \end{aligned} \quad (4.4)$$

in which  $f_0 := f(r_0)$  and  $\psi_0 = \psi(r_0)$ . Once more we placed the charge on the axis  $\theta = 0$ , and exploited the property  $Y_{\ell m}(0, \phi) = \sqrt{(2\ell+1)/(4\pi)} \delta_{m,0}$  of spherical-harmonic functions. The modes  $m \neq 0$  of the potential vanish.

Inserting the decomposition of Eq. (4.3) within Eq. (4.2) yields

$$F^r = e e^{-\psi_0} f_0 \lim_{x \rightarrow z} \sum_{\ell} \left[ (\partial_r \Phi_t)_\ell - (\partial_r \Phi_t^{\mathbf{S}})_\ell \right], \quad (4.5)$$

in which

$$(\partial_r \Phi_t)_\ell := \sum_{m=-\ell}^{\ell} \frac{d\Phi_{t\ell m}}{dr}(r) Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi}} \frac{d\Phi_{t\ell 0}}{dr}(r_0 + \Delta) \quad (4.6)$$

are the *multipole coefficients* of  $\partial_r \Phi_t$ , while  $(\partial_r \Phi_t^{\mathbf{S}})_\ell$  are those of the singular potential  $\Phi_t^{\mathbf{S}}$ . The limit  $x \rightarrow z$  is taken by setting  $r = r_0 + \Delta$ ,  $\theta = 0$ , and letting  $\Delta \rightarrow 0$ . The multipole coefficients can be obtained from the solution to Eq. (4.4). As in the scalar case, this equation must be integrated in the stellar interior of the polytropic model, and in the body's Schwarzschild exterior.

#### 4.2. Interior

The equation satisfied by the  $\Phi_{t\ell 0}$  mode of the electromagnetic potential was given above in Eq. (4.4), and since the charge is situated in the exterior portion of the spacetime ( $r_0 > R$ ), the right-hand side of the equation is zero everywhere within the interior. As in the scalar case we implement a change of variables from  $\Phi_{t\ell 0}$  to  $\eta_\ell$  defined by

$$\eta_\ell := \frac{d \ln \Phi_{t\ell 0}}{d \ln r}. \quad (4.7)$$

The equation for  $\Phi_{t\ell 0}$  becomes

$$\xi \frac{d\eta_\ell}{d\xi} + \eta_\ell(\eta_\ell - 1) + \mathcal{D}\eta_\ell - \frac{\ell(\ell+1)}{f} = 0, \quad (4.8)$$

with  $\mathcal{D} := 2 + \frac{1}{2} f^{-1} r (df/dr) - r (d\psi/dr)$  given explicitly by

$$\mathcal{D} = \frac{1}{f} \left\{ 2 - (n+1)b\xi^2 \left\{ 4\nu + [1 + (n+1)b\theta] \theta^n \right\} \right\}. \quad (4.9)$$

The equation is integrated outward from  $\xi = 0$ , at which  $\eta_\ell = \ell$ . Here also the numerical treatment is improved by adopting  $x = \ln \xi$  as new radial variable, and adding a few more terms to an expansion of  $\eta_\ell$  in powers of  $\xi^2$ . Integration proceeds until  $\xi = \xi_1$ , at which  $\eta_\ell = \eta_\ell^1$ .

#### 4.3. Exterior

Outside the matter the metric is described by the Schwarzschild solution, so that  $e^{2\psi} = f = 1 - 2M/r$ . Equation (4.4) becomes

$$r^2 f \frac{d^2 \Phi}{dr^2} + 2rf \frac{d\Phi}{dr} - \ell(\ell+1)\Phi = 4\pi(ee^{\psi_0}) \sqrt{\frac{2\ell+1}{4\pi}} f_0^{1/2} \delta(r-r_0), \quad (4.10)$$

where we suppress the label “ $t \ell 0$ ” on  $\Phi$ . The factor  $ee^{\psi_0}$  is equal to  $f_0^{1/2}$ , but we prefer to lump it with the electric charge  $e$  in order to keep the right-hand side in the same form as the right-hand side of Eq. (3.11); in this way the electromagnetic source is obtained directly from the scalar source by the replacement  $q \rightarrow -ee^{\psi_0}$ , with  $ee^{\psi_0}$  playing the role of an effective charge.

Once again it is helpful to use  $z := r/M - 1$  instead of  $r$  as a radial variable, and to re-express the differential equation as

$$(z^2 - 1) \frac{d^2 \Phi}{dz^2} + 2(z-1) \frac{d\Phi}{dz} - \ell(\ell+1)\Phi = 4\pi \frac{ee^{\psi_0}}{M} \sqrt{\frac{2\ell+1}{4\pi}} \left( \frac{z_0 - 1}{z_0 + 1} \right)^{1/2} \delta(z - z_0), \quad (4.11)$$

where  $z_0 := r_0/M - 1$ . Away from  $z_0$  the right-hand side of the equation vanishes, and the solution is a linear superposition of  $(z-1)P'_\ell(z)$  and  $(z-1)Q'_\ell(z)$ , in which a prime indicates differentiation with respect to  $z$ . The presence of the delta function implies that the solution must respect the junction conditions

$$[\Phi] = 0, \quad \left[ \frac{d\Phi}{dz} \right] = 4\pi \frac{ee^{\psi_0}}{M} \sqrt{\frac{2\ell+1}{4\pi}} \frac{1}{(z_0 - 1)^{1/2} (z_0 + 1)^{3/2}} \quad (4.12)$$

at  $z = z_0$ ; we recall that  $[f] := f(z = z_0 + \epsilon) - f(z = z_0 - \epsilon)$ , is the jump of the function  $f$  across  $z = z_0$ .

#### 4.4. Matching

The interior and exterior solutions must match smoothly at  $z = Z := R/M - 1$ , and the two exterior solutions must satisfy the junction conditions at  $z = z_0$ . For  $z < Z$  we have that  $\Phi = \Phi_{\text{in}}$ , for  $Z < z < z_0$  we have that  $\Phi = A(z-1)P'_\ell(z) + B(z-1)Q'_\ell(z)$ , where  $A$  and  $B$  are coefficients to be determined, and for  $z > z_0$  we have that  $\Phi = C(z-1)Q'_\ell(z)$ , where  $C$  is a third unknown coefficient. Letting  $\alpha := A(z_0 - 1)P'_\ell(z_0)$ ,  $\beta := B(z_0 - 1)Q'_\ell(z_0)$ ,  $\gamma := C(z_0 - 1)Q'_\ell(z_0)$ , and  $\delta := \Phi_{\text{in}}(Z)$ , we find that the matching conditions take the form of

$$\alpha = - \frac{(z_0 - 1)^2 (z_0 + 1)}{\ell(\ell+1)} P'_\ell(z_0) Q'_\ell(z_0) J, \quad (4.13)$$

$$\beta = \frac{(z_0 - 1)^2 (z_0 + 1)}{\ell(\ell+1)} \frac{\ell(\ell+1)P_\ell(Z) - (1 + \eta_\ell^1)(Z-1)P'_\ell(Z)}{\ell(\ell+1)Q_\ell(Z) - (1 + \eta_\ell^1)(Z-1)Q'_\ell(Z)} [Q'_\ell(z_0)]^2 J, \quad (4.14)$$

in which

$$J := -4\pi \frac{ee^{\psi_0}}{M} \sqrt{\frac{2\ell+1}{4\pi}} \frac{1}{(z_0 - 1)^{1/2} (z_0 + 1)^{1/2}}, \quad (4.15)$$

along with  $\gamma = \alpha + \beta$  and  $\delta = \{[(Z-1)P'_\ell(Z)]/[(z_0-1)P'_\ell(z_0)]\}\alpha + \{[(Z-1)Q'_\ell(Z)]/[(z_0-1)Q'_\ell(z_0)]\}\beta$ . As in the scalar case, the quantity of most direct interest is  $\gamma$ .

#### 4.5. Black-hole problem

If the electric charge were placed outside a black hole instead of in the vacuum region of a spherical star, the Schwarzschild metric would apply everywhere instead of being restricted to  $r > R$  or  $z > Z$ . In such a case the two external solutions would be given by  $\Phi = \bar{A}(z-1)P'_\ell(z)$  for  $z < z_0$ , and  $\Phi = \bar{C}(z-1)Q'_\ell(z)$  for  $z > z_0$ . Letting  $\bar{\alpha} := \bar{A}(z_0-1)P'_\ell(z_0)$  and  $\bar{\gamma} := \bar{C}(z_0-1)Q'_\ell(z_0)$ , we find that the junction conditions at  $z = z_0$  produce  $\bar{\alpha} = \bar{\gamma} = \alpha$ .

#### 4.6. Self-force difference

As in the scalar case we consider the difference between two self-forces,

$$\Delta F^r := F_{\text{star}}^r - F_{\text{hole}}^r = ee^{-\psi_0} f_0 \sum_{\ell} \left[ (\partial_r \Phi_{\text{star}})_\ell - (\partial_r \Phi_{\text{hole}})_\ell \right], \quad (4.16)$$

the first acting on a charge  $e$  at position  $r_0$  in the exterior of a material body, and the second acting on an identical charge at the same position outside a Schwarzschild black hole. Because the black-hole force is given by the Smith-Will expression  $F_{\text{hole}}^r = e^2 M f_0^{1/2} / r_0^3$  [6], we have that the stellar force is given by

$$F_{\text{star}}^r = \frac{e^2 M}{r_0^3} f_0^{1/2} + \Delta F^r. \quad (4.17)$$

Apart from a monopole piece that is entirely responsible for the black-hole force,  $\Phi_{\text{hole}}$  can be identified with the Detweiler-Whiting singular potential.

The force difference can be expressed as

$$\Delta F^r = ee^{-\psi_0} f_0 \sum_{\ell=0}^{\infty} \sqrt{\frac{2\ell+1}{4\pi}} \left[ \frac{d\Phi_{\text{star}}}{dr}(r_0 + \Delta) - \frac{d\Phi_{\text{hole}}}{dr}(r_0 + \Delta) \right], \quad (4.18)$$

in which each  $\Phi$  stands for the  $\ell 0$  mode of the spherical-harmonic expansion of the vector potential  $\Phi_\ell$ . The mode-sum requires the evaluation of  $d\Phi_{\text{star}}/dr$  and  $d\Phi_{\text{hole}}/dr$  at  $r = r_0 + \Delta$ . It is convenient to take  $\Delta = 0^+$ , so that  $r = r_0$  is approached from above. We insert the appropriate expressions within  $\Delta F^r$  and obtain

$$\Delta F^r = \frac{ee^{-\psi_0}}{M} \frac{f_0}{z_0 - 1} \sum_{\ell=1}^{\infty} \sqrt{\frac{2\ell+1}{4\pi}} (\gamma - \bar{\gamma}) \left[ -1 + \frac{\ell(\ell+1)}{z_0 - 1} \frac{Q_\ell(z_0)}{Q'_\ell(z_0)} \right], \quad (4.19)$$

in which a prime indicates differentiation with respect to  $z := r/M - 1$ ; the sum now excludes  $\ell = 0$ , because the monopole solutions are the same in the two spacetimes and therefore do not contribute to the difference. Inserting the result obtained previously for  $\gamma - \bar{\gamma} = \beta$ , we arrive at the expression of Eq. (1.2),

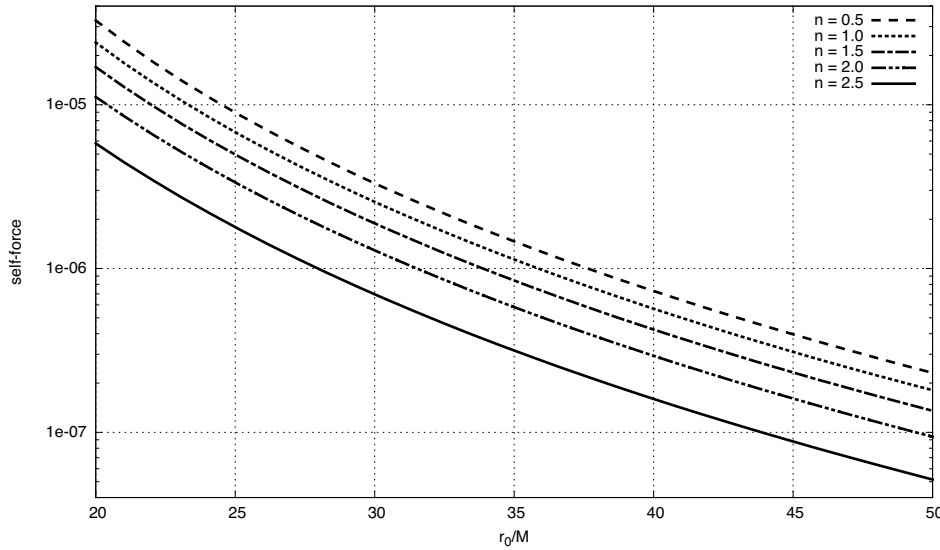
$$\Delta F^r = -\left(\frac{e}{M}\right)^2 \left(\frac{z_0 - 1}{z_0 + 1}\right)^{3/2} \sum_{\ell=1}^{\infty} (2\ell+1) S_\ell \left[ Q_\ell(z_0) - \frac{(z_0 - 1)Q'_\ell(z_0)}{\ell(\ell+1)} \right] Q'_\ell(z_0), \quad (4.20)$$

in which

$$S_\ell := \frac{\ell(\ell+1)P_\ell(Z) - (1 + \eta_\ell^1)(Z-1)P'_\ell(Z)}{\ell(\ell+1)Q_\ell(Z) - (1 + \eta_\ell^1)(Z-1)Q'_\ell(Z)} \quad (4.21)$$

is a structure factor that depends on the stellar model (through the stellar radius  $R$  and the interior constant  $\eta_\ell^1$ ) but is independent of  $r_0$ . We recall that  $z := r/M - 1$ ,  $Z := R/M - 1$ ,  $z_0 := r_0/M - 1$ , and that a prime indicates differentiation with respect





**Figure 6.** Electromagnetic self-force difference  $\Delta F^r$ , in units of  $(e/M)^2$ , plotted as a function of  $r_0/M$  for selected polytropic equations of state labelled by the polytropic index  $n$ . Each stellar model corresponds to a body of radius-to-mass ratio  $R/M \simeq 15$  (see Fig 2). The falloff with  $r_0$  is consistent with the asymptotic behaviour  $\sim r_0^{-5}$  predicted by Eq. (1.4), and for fixed  $r_0/M$  the self-force difference is seen to decrease with increasing  $n$ ; this behaviour is discussed and explained in Sec. 1.

to  $z$ . The mode-sum can be evaluated straightforwardly once the numerical results for  $\eta_\ell^1$  and  $R/M$  are available.

The self-force difference of Eq. (4.20) can be computed as a function of  $r_0$  for selected stellar models characterized by a polytropic index  $n$  and a relativistic parameter  $b$ . Sample results are presented in Fig. 6, and our findings were discussed more fully in Sec. 1. As in the scalar case the self-force for a massive thin shell can be computed by inserting  $\eta_\ell^1 = \ell/\sqrt{1 - 2M/R}$  within Eq. (3.24); this reproduces the results of Burko, Liu, and Soen [8].

## Acknowledgments

This work was supported by the Natural Sciences and Engineering Research Council of Canada, and by the Grant-in-Aid for the Global COE Program “The Next Generation of Physics, Spun from Universality and Emergence” from the Ministry of Education, Culture, Sports, Science and Technology of Japan. S. Isoyama is grateful to the Yukawa Institute for Theoretical Physics at Kyoto University for supporting his stay at University of Guelph. We thank Takahiro Tanaka for useful conversations that helped improve the paper.

## References

- [1] B. S. DeWitt and R. W. Brehme, *Radiation damping in a gravitational field*, Ann. Phys. (N.Y.) **9**, 220 (1960).

- [2] E. Poisson, A. Pound, and I. Vega, *The motion of point particles in curved spacetime*, Living Rev. Rel. **14**, 7 (2011), arXiv:1102.0529.
- [3] L. Barack and N. Sago, *Gravitational self-force on a particle in eccentric orbit around a Schwarzschild black hole*, Phys. Rev. **D81**, 084021 (2010), arXiv:1002.2386.
- [4] P. Diener, I. Vega, B. Wardell, and S. Detweiler, *Self-consistent orbital evolution of a particle around a Schwarzschild black hole* (2011), arXiv:1112.4821.
- [5] N. Warburton, S. Akcay, L. Barack, J. R. Gair, and N. Sago, *Evolution of inspiral orbits around a Schwarzschild black hole*, Phys. Rev. **D85**, 061501 (2012), arXiv:1111.6908.
- [6] A. G. Smith and C. M. Will, *Force on a static charge outside a Schwarzschild black hole*, Phys. Rev. D **22**, 1276 (1980).
- [7] A. G. Wiseman, *Self-force on a static scalar test charge outside a Schwarzschild black hole*, Phys. Rev. D **61**, 084014 (2000), arXiv:gr-qc/0001025.
- [8] L. M. Burko, Y. T. Liu, and Y. Soen, *Self-force on charges in the spacetime of spherical shells*, Phys. Rev. D **63**, 024015 (2001), arXiv:gr-qc/0008065.
- [9] W. Unruh, *Self force on charged particles*, Proc. Roy. Soc. London **A348**, 447 (1976).
- [10] K. Shankar and B. F. Whiting, *Self force of a static electric charge near a Schwarzschild Star*, Phys. Rev. **D76**, 124027 (2007), arXiv:0707.0042.
- [11] T. D. Drivas and S. E. Gralla, *Dependence of Self-force on Central Object*, Class. Quant. Grav. **28**, 145025 (2011), arXiv:1009.0504.
- [12] E. E. Flanagan and T. Hinderer, *Constraining neutron star tidal Love numbers with gravitational wave detectors*, Phys. Rev. D **77**, 021502(R) (2008), arXiv:0709.1915.
- [13] J. S. Read, C. Markakis, M. Shibata, K. Uryu, J. D. Creighton, and J. L. Friedman, *Measuring the neutron star equation of state with gravitational wave observations*, Phys. Rev. D **79**, 124033 (2009), arXiv:0901.3258.
- [14] T. Hinderer, B. D. Lackey, R. N. Lang, and J. S. Read, *Tidal deformability of neutron stars with realistic equations of state and their gravitational wave signatures in binary inspiral*, Phys. Rev. D **81**, 123016 (2010), arXiv:0911.3535.
- [15] F. Pannarale, L. Rezzolla, F. Ohme, and J. S. Read, *Will black hole-neutron star binary inspirals tell us about the neutron star equation of state?*, Phys. Rev. **D84**, 104017 (2011), arXiv:1103.3526.
- [16] T. Damour, A. Nagar, and L. Villain, *Measurability of the tidal polarizability of neutron stars in late-inspiral gravitational-wave signals* (2012), arXiv:1203.4352.
- [17] B. D. Lackey, K. Kyutoku, M. Shibata, P. R. Brady, and J. L. Friedman, *Extracting equation of state parameters from black hole-neutron star mergers. I. Nonspinning black holes*, Phys. Rev. **D85**, 044061 (2012), arXiv/1109.3402.
- [18] S. Detweiler and B. F. Whiting, *Self-force via a Green's function decomposition*, Phys. Rev. D **67**, 024025 (2003), arXiv:gr-qc/0202086.